Class 12 - §3.5 Composite functions

§3.6 Inverse functions

Review

$f(x) = \frac{1}{5}x - 5$, $f(5a-10) = \frac{1}{5}(5a-10) - 5 = \frac{5a}{5} - \frac{10}{5} - 5 = a - 2 - 5$

$= a - 7$.

$g(x) = \frac{7}{x-5}$, $g(a+2) = \frac{7}{a+2-5} = \frac{7}{a-3}$.

Simplify $\dfrac{2}{\frac{5}{x+8} + 8} = \dfrac{2}{\frac{5}{x+8} + \frac{8\cdot(x+8)}{1\cdot(x+8)}} = \dfrac{2}{\frac{5}{x+8} + \frac{8x+64}{x+8}}$

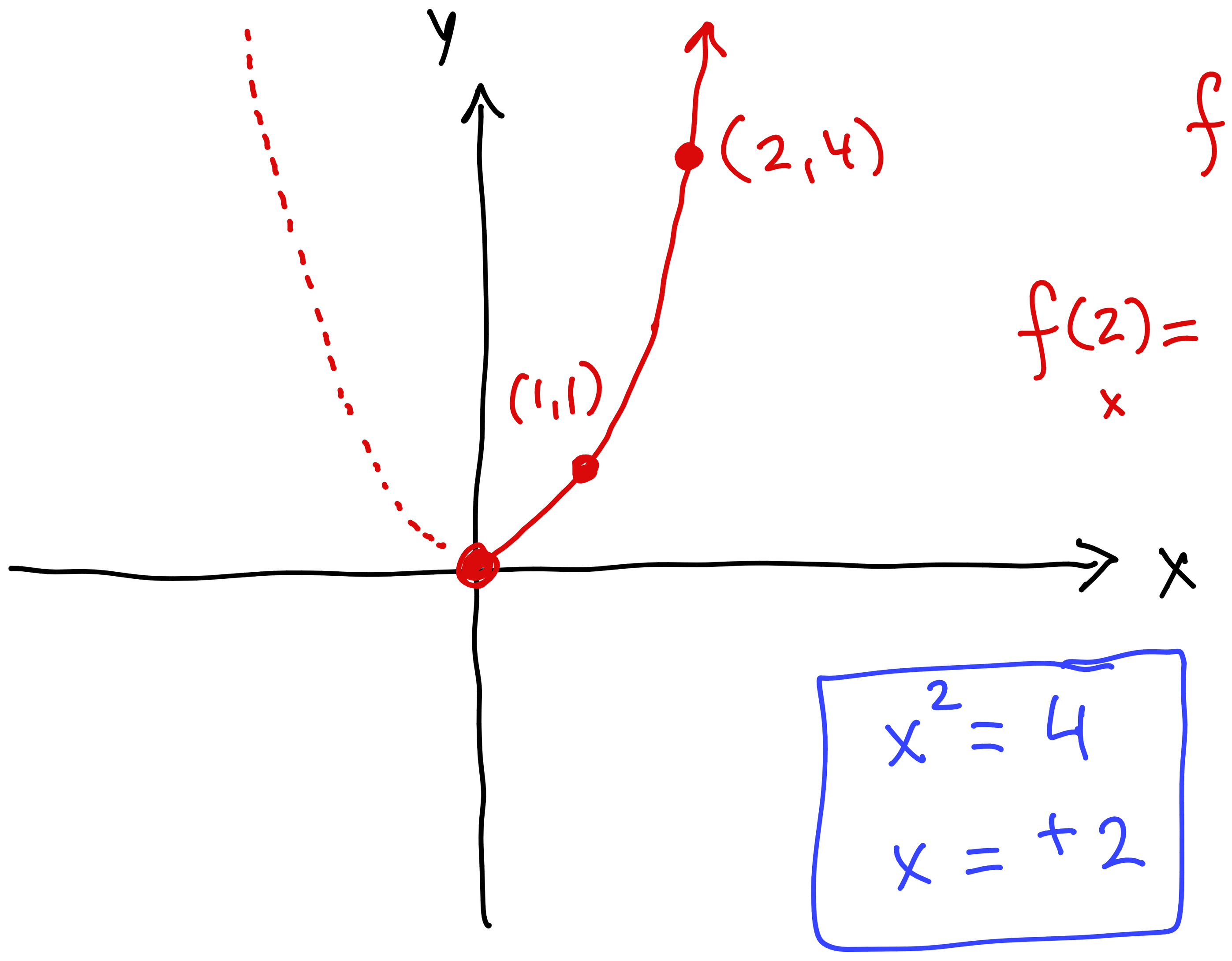
$= \dfrac{2}{\frac{5+8x+64}{x+8}} = \dfrac{2}{\frac{8x+69}{x+8}} = 2 \cdot \dfrac{x+8}{8x+69} = \dfrac{2(x+8)}{8x+69}$

$= \dfrac{2x+16}{8x+69}$.

Simplify $\dfrac{3x-9}{\frac{2x+1}{x-1} - \frac{3}{1}\frac{(x-1)}{(x-1)}} = \dfrac{3x-9}{\frac{2x+1}{x-1} - \frac{3x-3}{x-1}} = \dfrac{3x-9}{\frac{2x+1-3x+3}{x-1}}$

$= \dfrac{3x-9}{\frac{-x+4}{x-1}} = \dfrac{3x-9}{1} \cdot \dfrac{x-1}{-x+4} = \dfrac{(3x-9)(x-1)}{-x+4}$.

Ex1: Graph $f(x) = x^2$, $x \geq 0$:



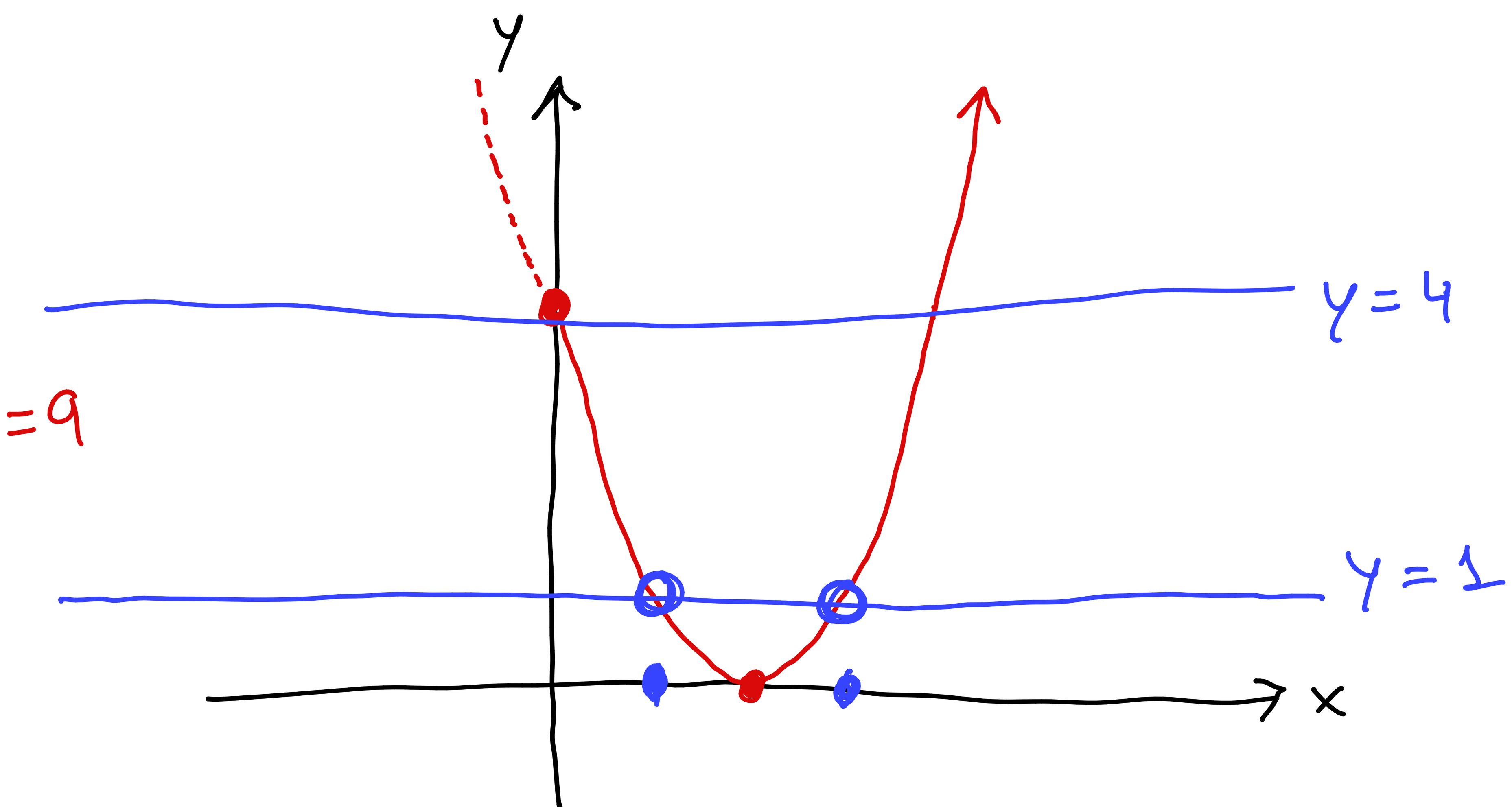
$f(1) = 1$ (x, y)

$f(2) = 4$ (x, y)

$f(3) = 9$

$$\boxed{\begin{array}{l} x^2 = 4 \\ x = \pm 2 \end{array}}$$

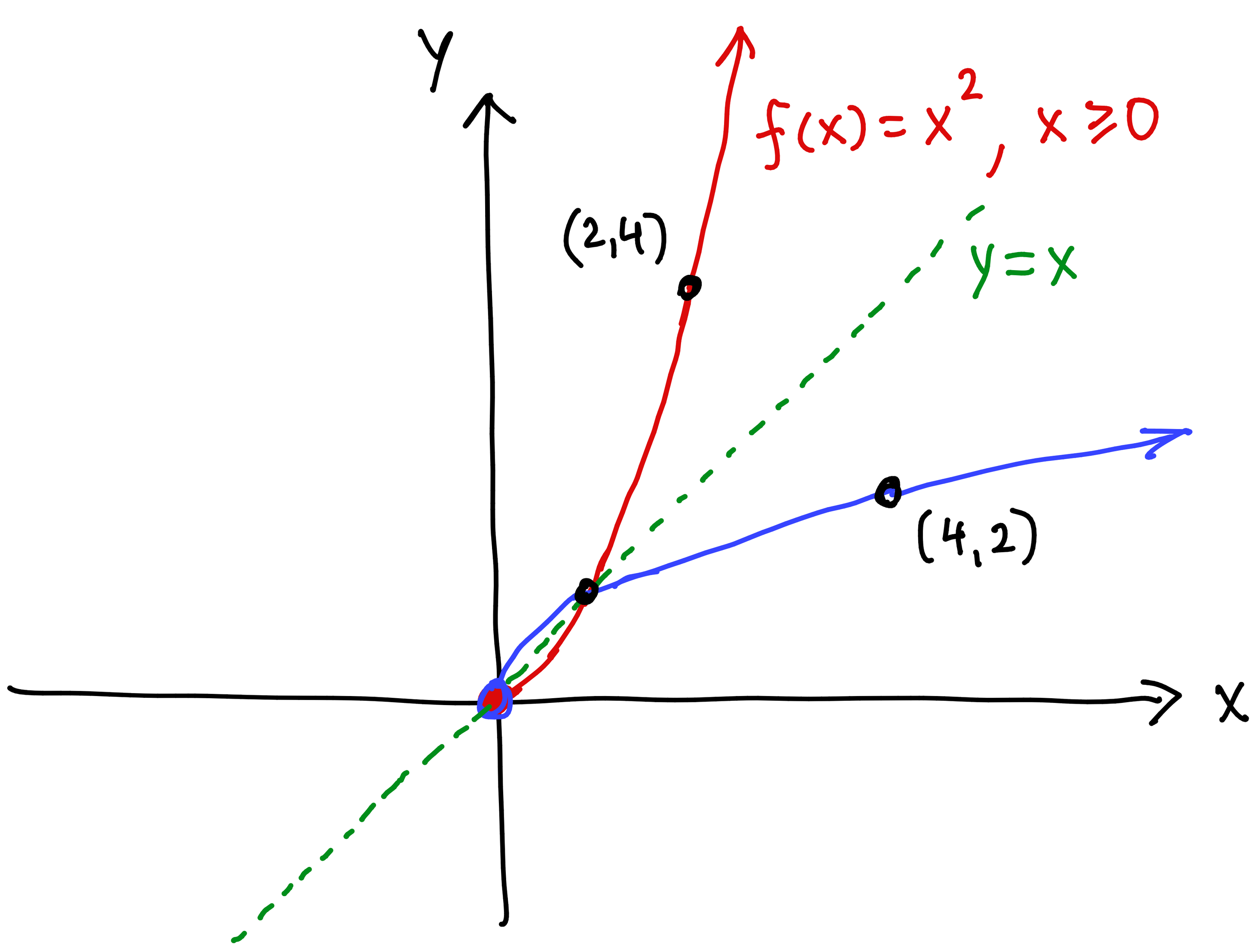
Ex2: Graph $g(x) = (x-2)^2$, $x \geq 0$:



Not 1 to 1

$\sqrt{(x-2)^2} = \sqrt{1} \Rightarrow x - 2 = \pm 1 \Leftrightarrow x = 2 \pm 1 \Leftrightarrow x = 1$ or $x = 3$

The horizontal line test: identify when a function is 1-to-1.



$g(x) = \sqrt{x}$

Ex 3: Verify if the following function is 1-to-1.

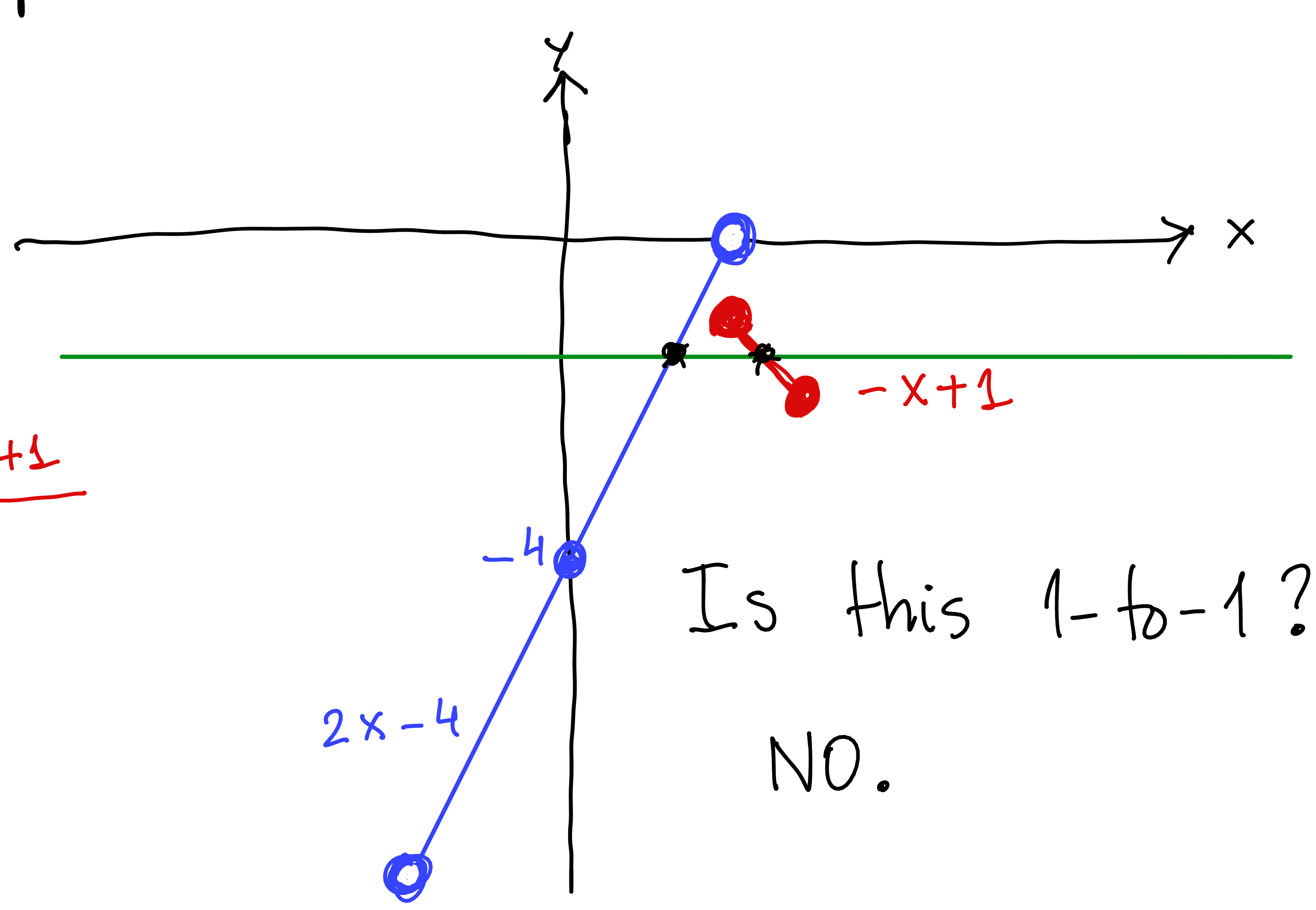
$$f(x) = \begin{cases} 2x - 4, & \text{if } -2 < x < 2 \quad (1) \\ -x + 1, & \text{if } 2 \leq x \leq 3 \quad (2) \end{cases}$$

$y = x \to 2x \to 2x - 4$

$-x + 1 = -(x - 1)$

x-ref. id 1 down 1 right

x	y=-x+1
0	1
1	0
2	-1



Is this 1-to-1?

NO.

Solve for y:

a) $2(x) = \left(\frac{3y-9}{\cancel{2}}\right)\cancel{2} \Rightarrow 2x = 3y - \cancel{9} \iff \frac{2x+9}{3} = \frac{\cancel{3}y}{\cancel{3}} \iff \boxed{\frac{2x+9}{3} = y}.$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad +9 \quad\quad +\cancel{9}$

b) $x = -\sqrt[3]{y-2} + 6 \iff (x-6)^3 = \left(-\sqrt[3]{y-2}\right)^3 \iff (x-6)^3 = -(y-2)$

$\quad -6 \quad\quad\quad\quad\quad -6$

$\iff (x-6)^3 = -y + \cancel{2} \iff \frac{(x-6)^3 - 2}{-1} = \frac{-y}{-1} \iff \boxed{-(x-6)^3 + 2 = y}.$

$\quad\quad -2 \quad\quad\quad -\cancel{2}$

c) $(x)3 = \left(\frac{6y+1}{\cancel{3}}\right)\cancel{3} \iff 3x = 6y + 1 \iff \frac{3x-1}{6} = \frac{\cancel{6}y}{\cancel{6}} \iff \boxed{y = \frac{3x-1}{6}}.$

$\quad\quad\quad\quad\quad\quad\quad\quad -1 \quad\quad -1$

Composite functions:

$f(x) = 7x+9$ & $g(x) = \frac{7}{x+9}$. $\boxed{(f \circ g)(x) = f(g(x))}$

Compute $(f \circ g)(x)$:

$(f \circ g)(x) = f(g(x)) = 7g(x) + 9 = 7\left(\frac{7}{x+9}\right) + 9 = \frac{49}{x+9} + 9\,\frac{(x+9)}{(x+9)}$

$= \frac{49 + 9x + 81}{x+9} = \frac{9x+130}{x+9}.$

$\boxed{g(1) = \frac{7}{1+9} = \frac{7}{10}}$

$\boxed{(f \circ g)(x) = \frac{9x+130}{x+9}}$

• $(f \circ g)(1) = \frac{9 \cdot 1 + 130}{1+9} = \frac{139}{10}.$

$(f \circ g)(1) = f(g(1)) = f\left(\frac{7}{10}\right) = 7 \cdot \left(\frac{7}{10}\right) + 9$

$= \frac{49}{10} + \frac{9 \cdot 10}{1 \cdot 10} = \frac{49}{10} + \frac{90}{10} = \frac{139}{10}.$

What is $(g \circ f)(x)$?

$f(x) = 7x+9$

$g(x) = \frac{7}{x+9}$

$(g \circ f)(x) = g(f(x)) = \frac{7}{f(x)+9} = \frac{7}{7x+9+9}$

$\boxed{(g \circ f)(x) = \frac{7}{7x+18}}$

Two functions $f(x)$ & $g(x)$ are inverses iff $(g \circ f)(x) = x$ and
$(f \circ g)(x) = x$

Ex1: $f(x) = x^2, x \geq 0$ & $g(x) = \sqrt{x}$

$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (\sqrt{x})^2 = x.$

$(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{x^2} = x.$

Both $(f \circ g)(x) = x$ & $(g \circ f)(x) = x$. We conclude g and f are inverses.

Remark: We represent the inverse function as $f^{-1}(x)$.

$f^{-1}(x) \neq \frac{1}{f(x)}$. In ex1, if $f(x) = x^2, x \geq 0$, then $f^{-1}(x) = \sqrt{x}$.

Ex2: Show that $f(x) = 2x - 3$ and $g(x) = \frac{x+3}{2}$ are inverses.

1: compute $(g \circ f)(x)$ & $(f \circ g)(x)$

2) if $(g \circ f)(x) = x$ & $(f \circ g)(x) = x$, then f and g are inverses.

$(f \circ g)(x) = f(g(x)) = 2(g(x)) - 3 = \cancel{2} \cdot \left(\frac{x+3}{\cancel{2}}\right) - 3 = x + 3 - 3 = x.$

$(g \circ f)(x) = g(f(x)) = \frac{f(x) + 3}{2} = \frac{(2x \cancel{-3}) \cancel{+3}}{2} = \frac{2x}{2} = x.$

So f and g are inverses. $f(2) = 1$ & $g(1) = 2$

Finding the inverse:

Ex 1: $f(x) = \dfrac{2x+5}{3}$. Goal: find $f^{-1}(x)$.

Step 0: Check if $f(x)$ is 1-to-1.

Step 1: replace $f(x) = y$: $\quad y = \dfrac{2x+5}{3}$.

Step 2: switch x & y: $\quad x = \dfrac{2y+5}{3}$.

Step 3: solve for y: $\quad y = \dfrac{3x-5}{2}$.

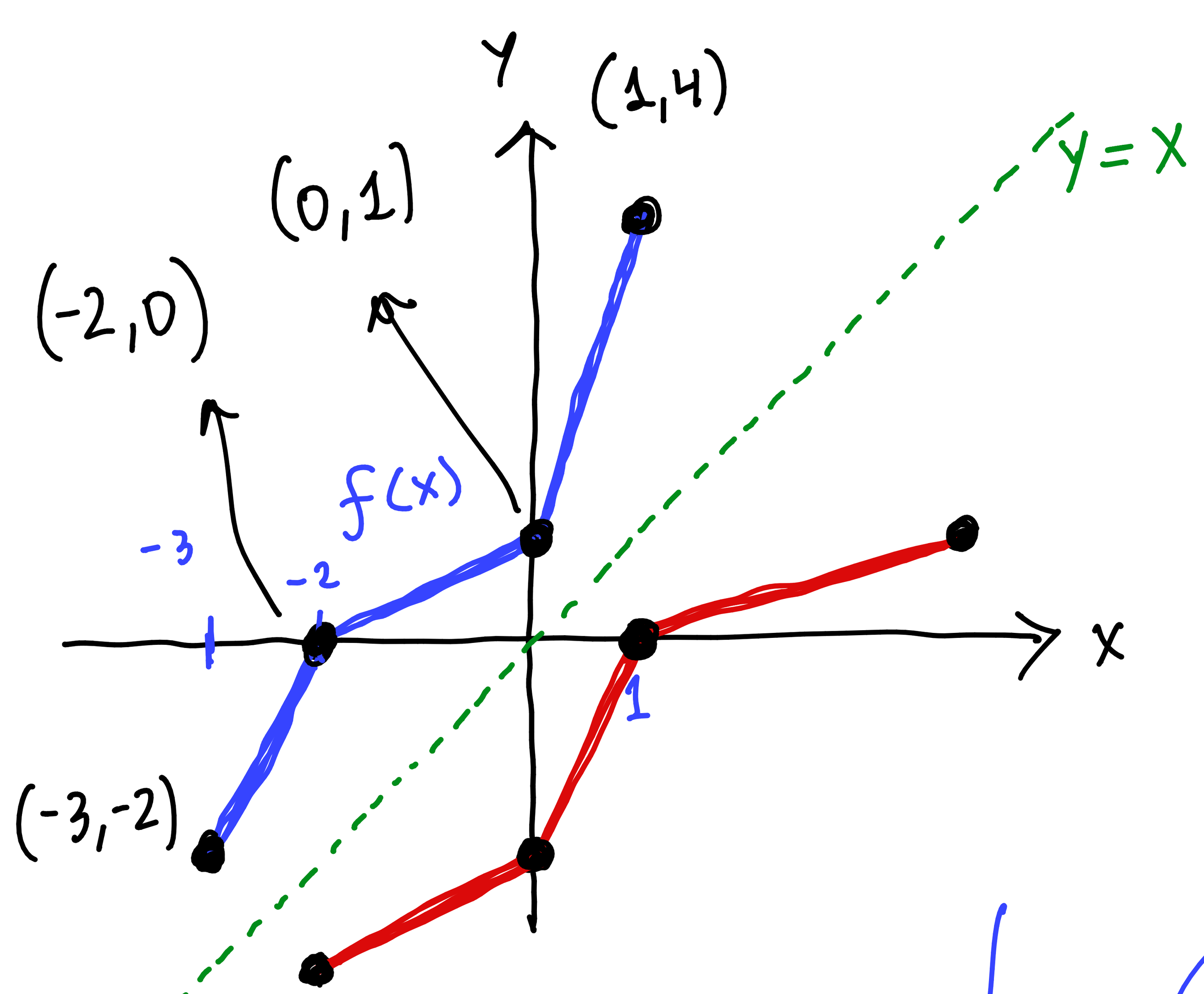
Step 4: replace $y = f^{-1}(x)$: $\quad f^{-1}(x) = \dfrac{3x-5}{2}$

$$\boxed{\begin{aligned} x\cdot 3 &= \left(\frac{2y+5}{\cancel{3}}\right)\cdot\cancel{3} \\ 3x &= 2y+5 \\ -5 &\qquad\quad -5 \\ \frac{3x-5}{2} &= \frac{2y}{2} \end{aligned}}$$

Ex 2: Find the inverse of $f(x) = \frac{1}{9}x - 2$.

$$\underset{\text{step 1}}{y = \tfrac{1}{9}x - 2} \;\rightarrow\; \underset{\text{step 2}}{x = \tfrac{1}{9}y - 2} \;\rightarrow\; \underset{\text{step 3}}{y = 9x + 18} \;\rightarrow\; \underset{\text{step 4.}}{f^{-1}(x) = 9x + 18}$$

Ex 3: Find the inverse of $f(x)$ using the graphs:



graph = set of all points of the type $(x, f(x))$

$f(-3) = -2 \;\leftrightarrow\; (-3, f(-3))$
$f(-2) = 0$
$f(0) = 1$
$f(1) = 4$

$f^{-1}(-2) = -3 \leftarrow (-2,-3)$
$f^{-1}(0) = -2 \leftarrow (0,-2)$
$f^{-1}(1) = 0 \leftarrow (1,0)$
$f^{-1}(4) = 1 \leftarrow (4,1)$